

# A new theoretical approach to Interacting Dimers

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Recent numerical results on classical dimers with aligning interactions have been theoretically explained via a Coulomb Gas representation of the height random variable. Here we propose a different, more direct, theoretical viewpoint, the Interacting Fermions Picture, which provides a better account of the large-distance asymptotic formulas of certain correlations.

## I. INTRODUCTION

The lattice model of hard-core close-packed dimers is among the most fundamental systems in two-dimensional Statistical Mechanics. Not only it is exactly solvable in a broad sense (the free energy and many correlations can be computed, [1–5]) but it also provides, through equivalences, the exact solutions of several other models, including the nearest neighbor Ising model, [1, 6], and some vertex models at the so called *free fermion point*, [7, 8].

Recently, especially in connection with a problem of Quantum Statistical Mechanics, [9], several authors have been studying the (classical) dimer model on a square lattice, modified with a nearest neighbor aligning interaction. We will call it *interacting dimer model*, (IDM). Since a general exact solution for the IDM has not been found so far, our knowledge of the properties of this system - quite a deep knowledge, in fact - rests entirely on the numerical analysis of [10–13]; and on their theoretical interpretation via the *Coulomb Gas approach*, (CGA).

The CGA, [14–16] (see also [17]’s review), applies every time in the model there is a natural *height random variable*; and it is based of the postulate that, in the scaling limit, height variable correlations are equal to charge correlations of a free boson field. From a practical viewpoint, the CGA has been very successful in describing the critical exponents of virtually *every* critical model of two-dimensional, lattice Statistical Mechanics. Though, the conjectured scaling limit of the height variable has been difficult to substantiate (the best result in this direction is still nowadays [18]). Besides, in the case of the IDM, the CGA has not provided so far a clear account of some fine details, such as the staggered prefactors of the correlations (see below).

This article proposes a different theoretical approach to the IDM, that can be called *Interacting Fermions Picture* (IFP). This approach is not new in Physics. It has been a standard tool in condensed matter theory for studying the 1+1-dimensional quantum models (see reviews [19, 20]). In classical 2-dimensional Statistical Mechanics, it was first employed in [21] to demonstrate the universality of the nearest neighbors Ising model under small, “solvability breaking”, perturbations; and recently it was used to study the weak-universal properties of the Eight-Vertex and the Ashkin-Teller models, [22, 23]. In regard to the IDM, fermion viewpoints have already been employed in [24, 25]; however, the full fledged IFP application to the IDM that is given here appears to be

completely new. The method is made of two steps: i) the IDM is re-casted into a lattice fermion field with a self-interactions; ii) the scaling limit of the lattice field is proved to be the Thirring model, which is interacting as well, but which is also exactly solvable. The IFP solves some difficulties of the CGA because: a) the convergence, in the scaling limit, of lattice correlations to the Thirring model correlations naturally emerges from renormalization group ideas; b) it makes clear the origin of the staggering prefactors of the dimer correlations. Besides, via *bosonization* of the Thirring model, one could always recover the free boson description that is typical of the CGA, without ever introducing the height variable of the lattice model.

In our view, the IFP is a most natural and transparent way to describe the IDM.

## II. DEFINITIONS AND RESULTS

Consider a finite box  $\Lambda$  of the infinite square lattice. A *dimer configuration*,  $\omega$ , is a collection of dimers covering the edges of  $\Lambda$  with the constraint that every vertex of  $\Lambda$  is covered by one, and only one, dimer. The partition function of the interacting dimers model (IDM) is

$$Z_\lambda(\Lambda) = \sum_{\omega} e^{\lambda \sum_{d,d' \in \omega} v(d,d')} \quad (1)$$

where:  $\lambda$  is a real parameter;  $v(d,d')$  is a two body dimer interaction; the first sum is over all the dimer configurations; the second sum is over any pair of dimers in the configuration  $\omega$ . In [11]  $\lambda v(d,d')$  is a nearest neighbor aligning interaction. In this work we assume more in general that  $v(d,d')$  is zero unless  $d$  and  $d'$  are both horizontal or both vertical; that it is invariant under  $\pi/2$ -rotations and under lattice translations; and that  $|v(d,d')|$  has exponential decay in the distance between  $d$  and  $d'$ . The non-interacting, exactly solvable, dimer model is the case  $\lambda = 0$ , [1–5].

Our main result is the evaluation of correlation critical exponents of local bulk observables, for small  $|\lambda|$ . A natural observable to consider is the *dimer occupancy*  $\nu_d(\omega)$ , which is equal to 1 if the dimer  $d$  is present in  $\omega$ , and it is zero otherwise. Consider the horizontal dimers  $d = \{0, \mathbf{e}_0\}$  and  $d' = \{\mathbf{x}, \mathbf{x} + \mathbf{e}_0\}$ , for  $\mathbf{e}_0 = (1, 0)$  and

$\mathbf{x} = (x_0, x_1)$ . The IFP provides the large- $|\mathbf{x}|$  formula

$$\langle \nu_d \nu_{d'} \rangle - \langle \nu_d \rangle \langle \nu_{d'} \rangle \sim (-1)^{x_0+x_1} c \frac{x_0^2 - x_1^2}{(x_0^2 + x_1^2)^2} + (-1)^{x_0} c_- \frac{1}{(x_0^2 + x_1^2)^{\kappa_-}} \quad (2)$$

for *critical exponent*  $\kappa_- = 1 + O(\lambda v)$  (and for  $\lambda v$ -dependent  $c$  and  $c_-$ ). Notice that according to the sign of  $\lambda$  either the former or the latter term in (2) is the leading one at large  $|\mathbf{x}|$ . (2) coincides with the exact solution for  $\lambda = 0$ : see (7.12) and (7.20) of [3]; and it is in agreement with the numerical simulations for  $\lambda > 0$  (and smaller than a threshold value): see (51) and (52) of [11].

The critical exponents  $\kappa_-$  is non-universal, because it does depend upon  $\lambda v$ . What is expected to be universal, instead, is the relationship among critical exponents of different observables. It is instructive to study a second observable, then. The authors of [11] considered the monomer correlation. We could describe this correlation in the IFP as well; however, being the monomer correlation equivalent to a *non-local* fermion correlation, the derivation of the scaling limit in the IFP, at the present time, is not more transparent than in the CGA.

We consider instead a different observable, the *diagonal dimer*. It consists in a pair of monomers in positions  $\{\mathbf{x}, \mathbf{x} + \mathbf{e}\}$ , where  $\mathbf{e} = (1, 1)$ ; see Fig. 1. Since such a dimer is not allowed in the hard-core, close-packed configurations  $\omega$ , we define its “correlation” as it is done for the monomer observable, i.e. in terms of lattice defects:

$$\langle \nu_d \rangle = \lim_{\Lambda \rightarrow \infty} \frac{Z_\Lambda(\Lambda - d)}{Z_\Lambda(\Lambda)} \quad \langle \nu_d \nu_{d'} \rangle = \lim_{\Lambda \rightarrow \infty} \frac{Z_\Lambda(\Lambda - (d \cup d'))}{Z_\Lambda(\Lambda)}.$$

For  $d = \{0, \mathbf{e}\}$  and  $d' = \{\mathbf{x}, \mathbf{x} + \mathbf{e}\}$ , the IFP gives the large- $|\mathbf{x}|$  formula

$$\langle \nu_d \nu_{d'} \rangle - \langle \nu_d \rangle \langle \nu_{d'} \rangle \sim c_+ \frac{(-1)^{x_0} - (-1)^{x_1}}{(x_0^2 + x_1^2)^{\kappa_+}} \quad (3)$$

for a new *critical exponent*  $\kappa_+ = 1 + O(\lambda v)$  (and for a  $\lambda v$ -dependent  $c_+$ ). The universal formula that relates  $\kappa_+$  to  $\kappa_-$  is peculiar of the models with central charge  $c = 1$  and was originally discovered by Kadanoff, [26]:

$$\kappa_+ \cdot \kappa_- = 1. \quad (4)$$

In the next section we introduce the fermion representation of the IDM and derive (2), (3) and (4). In fact, a power series expansion in  $\lambda$  for  $\kappa_-$  and  $\kappa_+$  could also be provided, but we will not do it here (see [23]). We remark that the CGA was successfully used to justify the appearance of critical exponents  $\kappa_-$  and  $\kappa_+$  which satisfy (4) (see pt.1 and pt.5 on page 7 of [27]); however, in that approach, the derivations of the staggering prefactors of (2) and (3), as well as of the power series formulas for  $\kappa_-$  and  $\kappa_+$ , are *not* available.

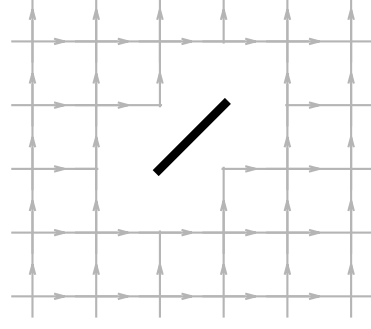


FIG. 1: Example of a diagonal dimer. Its presence in the graph generate a new face with: 6 vertical arrows, 3 of which are clockwise oriented; 6 horizontal arrows, 3 of which are clockwise oriented. The total oddness of this face is then  $i^3(-i)^3 1^3(-1)^3 = -1$  and so, as opposed to the case of the monomer defect, [3], the fermion representation of the diagonal dimer remains local.

### III. INTERACTING FERMIONS PICTURE

When  $\lambda = 0$  the dimer model is equivalent to a lattice fermion field without interaction. Namely

$$Z_0(\Lambda) = \int D\psi \exp \left\{ -\frac{1}{2} \sum_{\mathbf{x}, \mathbf{y}} K_{\mathbf{x}, \mathbf{y}} \psi_{\mathbf{x}} \psi_{\mathbf{y}} \right\} \quad (5)$$

where:  $\{\psi_{\mathbf{x}} : \mathbf{x} \in \Lambda\}$  are Grassmann variables and  $D\psi$  indicates the integration w.r.t. all of them;  $K_{\mathbf{x}, \mathbf{y}}$  is the *Kasteleyn matrix* that can be chosen to be such that

$$\sum_{\mathbf{y}} K_{\mathbf{x}, \mathbf{y}} \psi_{\mathbf{y}} = \sum_{\sigma=\pm 1} \sigma (\psi_{\mathbf{x}+\sigma \mathbf{e}_0} + i \psi_{\mathbf{x}+\sigma \mathbf{e}_1})$$

with  $\mathbf{e}_0 = (1, 0)$  and  $\mathbf{e}_1 = (0, 1)$ . (5) is the partition function of a free *Majorana fermion field*, i.e. a Grassmann-valued Gaussian field with moment generator

$$\langle e^{\sum_{\mathbf{x}} \psi_{\mathbf{x}} \eta_{\mathbf{x}}} \rangle_0 = e^{-\frac{1}{2} \sum_{\mathbf{x}, \mathbf{y}} S(\mathbf{x}-\mathbf{y}) \eta_{\mathbf{x}} \eta_{\mathbf{y}}}$$

where:  $\{\eta_{\mathbf{x}} : \mathbf{x} \in \mathbb{Z}^2\}$  are other Grassmann variables;  $S$ , the covariance, is the inverse Kasteleyn matrix

$$S(\mathbf{x}) = \langle \psi_{\mathbf{x}} \psi_0 \rangle_0 = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dp_0}{2\pi} \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \frac{e^{ip_0 x_0 + ip_1 x_1}}{i \sin p_0 - \sin p_1}.$$

The Fourier transform of  $S$  is singular at four *Fermi momenta*:  $\mathbf{p}_{+,0} = (0, 0)$ ,  $\mathbf{p}_{+,1} = (\pi, \pi)$ ,  $\mathbf{p}_{-,0} = (0, \pi)$  and  $\mathbf{p}_{-,1} = (\pi, 0)$ . Therefore, in view of the scaling limit, it is convenient to decompose

$$\psi_{\mathbf{x}} = \sum_{\substack{\omega=\pm \\ s=0,1}} i^s e^{i\mathbf{p}_{\omega,s} \cdot \mathbf{x}} \psi_{\mathbf{x},\omega,s}$$

where  $\psi_{\mathbf{x},\omega,s}$  are four *independent* Majorana fields with large- $|\mathbf{x}|$  covariances

$$\begin{aligned} \langle \psi_{\mathbf{x},\omega,s} \psi_{0,\omega,s} \rangle_0 &\sim \frac{1}{2} \int \frac{dp_0}{2\pi} \int \frac{dp_1}{2\pi} \frac{e^{ip_0 x_0 + ip_1 x_1}}{ip_0 - \omega p_1} \\ &= \frac{1}{4\pi} \frac{1}{x_0 + i\omega x_1}. \end{aligned} \quad (6)$$

This decomposition already appeared in [24]. Anyways here we are preparing for the application to the interacting case, therefore we also introduce the Dirac spinors  $\psi_{\mathbf{x}}^+ = (\psi_{\mathbf{x},+}^+, \psi_{\mathbf{x},-}^+)$  and  $\psi_{\mathbf{x}} = (\psi_{\mathbf{x},+}, \psi_{\mathbf{x},-})^T$  for

$$\psi_{\mathbf{x},\omega}^+ = \frac{\psi_{\mathbf{x},\omega,0} + i\psi_{\mathbf{x},\omega,1}}{\sqrt{2}} \quad \psi_{\mathbf{x},\omega} = \frac{\psi_{\mathbf{x},\omega,0} - i\psi_{\mathbf{x},\omega,1}}{\sqrt{2}} \quad (7)$$

with translational invariant covariances

$$\begin{aligned} \langle \psi_{\mathbf{x},\omega} \psi_{0,\omega'} \rangle_0 &= 0, & \langle \psi_{\mathbf{x},\omega}^+ \psi_{0,\omega'}^+ \rangle_0 &= 0 \\ \langle \psi_{\mathbf{x},\omega}^+ \psi_{0,\omega'} \rangle_0 &\sim \frac{\delta_{\omega,\omega'}}{4\pi} \frac{1}{x_0 + i\omega x_1}. \end{aligned} \quad (8)$$

If we now let  $\lambda \neq 0$ , by power series expansion in  $\lambda$  one can verify (by *cluster expansion*) that (1) becomes

$$Z_\lambda(\Lambda) = \int D\psi \exp \left\{ -\frac{1}{2} \sum_{\mathbf{x},\mathbf{y}} K_{\mathbf{x},\mathbf{y}} \psi_{\mathbf{x}} \psi_{\mathbf{y}} + V(\lambda v, \psi) \right\} \quad (9)$$

where  $V(\lambda v, \psi)$  is an even polynomial in the  $\psi$ . In the next section, by a Renormalization Group argument, we will explain why, in the evaluation of the large distance decay of the correlations, it is correct to replace the interacting fermion field (9) with the Thirring model. Assuming for the moment this crucial fact, we now compute the correlation critical exponents. It is not difficult to see that the dimer correlation in the l.h.s. of (2) becomes, in terms of the Dirac fermions (7) (and up to terms with faster decays)

$$\begin{aligned} &\langle \psi_0 \psi_{\mathbf{e}_0} \psi_{\mathbf{x}} \psi_{\mathbf{x}+\mathbf{e}_0} \rangle - \langle \psi_0 \psi_{\mathbf{e}_0} \rangle \langle \psi_{\mathbf{x}} \psi_{\mathbf{x}+\mathbf{e}_0} \rangle \\ &\sim 4(-1)^{x_0+x_1} \sum_{\omega} \langle \psi_{0,\omega}^+ \psi_{0,\omega}; \psi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega} \rangle^T \\ &+ 4(-1)^{x_0} \sum_{\omega} \langle \psi_{0,\omega}^+ \psi_{0,-\omega}; \psi_{\mathbf{x},-\omega}^+ \psi_{\mathbf{x},\omega} \rangle^T \end{aligned} \quad (10)$$

where the label  $T$  indicates a truncated correlations. In the same way, the diagonal dimer correlation in the l.h.s. of (3) becomes

$$\begin{aligned} &\langle \psi_0 \psi_{\mathbf{e}} \psi_{\mathbf{x}} \psi_{\mathbf{x}+\mathbf{e}} \rangle - \langle \psi_0 \psi_{\mathbf{e}} \rangle \langle \psi_{\mathbf{x}} \psi_{\mathbf{x}+\mathbf{e}} \rangle \\ &\sim 8[(-1)^{x_0} - (-1)^{x_1}] \langle \psi_{0,+}^+ \psi_{0,-}^+; \psi_{\mathbf{x},-} \psi_{\mathbf{x},+} \rangle^T \end{aligned} \quad (11)$$

The last step is to look up the exact solution for the massless Thirring model correlations, [28–30] (see also the review [31]). We find

$$\begin{aligned} \langle \psi_{\omega}^{\dagger}(0) \psi_{\omega}(0); \psi_{\omega}^{\dagger}(\mathbf{x}) \psi_{\omega}(\mathbf{x}) \rangle^T &= c \frac{x_0^2 - x_1^2}{(x_0^2 + x_1^2)^2} \\ \langle \psi_{+}^{\dagger}(0) \psi_{-}^{\dagger}(0); \psi_{-}(\mathbf{x}) \psi_{+}(\mathbf{x}) \rangle^T &= \frac{c_{-}}{(x_0^2 + x_1^2)^{\kappa_{+}}} \\ \langle \psi_{\omega}^{\dagger}(0) \psi_{-\omega}(0); \psi_{-\omega}^{\dagger}(\mathbf{x}) \psi_{\omega}(\mathbf{x}) \rangle^T &= \frac{c_{+}}{(x_0^2 + x_1^2)^{\kappa_{-}}} \end{aligned} \quad (12)$$

for critical exponents

$$\kappa_{+} = \frac{1 + \frac{\lambda_T}{4\pi}}{1 - \frac{\lambda_T}{4\pi}} \quad \kappa_{-} = \frac{1 - \frac{\lambda_T}{4\pi}}{1 + \frac{\lambda_T}{4\pi}}$$

where  $\lambda_T = O(\lambda v)$  is a parameter of the Thirring model. The derivation of (2), (3), (4) is complete.

## IV. RG ANALYSIS

We follow the Wilson's RG scheme in the version due to Gallavotti, [32]. Integrating out the large momentum scales, one obtains an effective interaction

$$\begin{aligned} &\sum_n \sum_{\substack{\omega_1, \dots, \omega_n \\ s_1, \dots, s_{2n}}} i^{s_1 + \dots + s_{2n}} \int d\mathbf{k}_1 \dots d\mathbf{k}_{2n} \hat{\psi}_{\mathbf{k}_1, \omega_1, s_1} \dots \hat{\psi}_{\mathbf{k}_{2n}, \omega_{2n}, s_{2n}} \\ &\cdot \delta \left( \sum_{j=1}^{2n} \mathbf{k}_j + \sum_{j=1}^{2n} \mathbf{p}_{\omega_j, s_j} \right) \hat{w}_{2n}(\mathbf{k}_2 + \mathbf{p}_{\omega_2, s_2}, \dots, \mathbf{k}_{2n} + \mathbf{p}_{\omega_{2n}, s_{2n}}). \end{aligned}$$

It is important to take into account some symmetries. For  $R(k_0, k_1) = (k_1, -k_0)$ ,  $\vartheta(k_0, k_1) = (k_1, k_0)$  and  $\tau(k_0, k_1) = (k_0, k_1 + \pi)$ , we find

$$\begin{aligned} \hat{w}_{2m}(\tau \mathbf{k}_2, \dots, \tau \mathbf{k}_{2m}) &= (-i)^m \hat{w}_{2m}(\vartheta \mathbf{k}_2, \dots, \vartheta \mathbf{k}_{2m}) \\ \hat{w}_{2m}(R \mathbf{k}_2, \dots, R \mathbf{k}_{2m}) &= (-i)^m \hat{w}_{2m}(\mathbf{k}_2, \dots, \mathbf{k}_{2m}) \\ \hat{w}_{2m}(\mathbf{k}_2, \dots, \mathbf{k}_{2m})^* &= i^m \hat{w}_{2m}(\vartheta \mathbf{k}_2, \dots, \vartheta \mathbf{k}_{2m}). \end{aligned} \quad (13)$$

From power counting, there are two possible local, marginal terms: a quartic term, that requires the renormalization of the coupling constant  $\lambda$ ; a quadratic term, responsible for a field renormalization. By (13) they are:

$$4\hat{w}_4(\mathbf{p}_{+,1}, \mathbf{p}_{-,0}, \mathbf{p}_{-,1}) \sum_{\mathbf{x}} \psi_{\mathbf{x},+}^+ \psi_{\mathbf{x},+}^- \psi_{\mathbf{x},-}^+ \psi_{\mathbf{x},-}^- \quad (14)$$

and

$$2 \left[ -i \frac{\partial \hat{w}_2}{\partial k_0}(\mathbf{p}_{+,0}) - \frac{\partial \hat{w}_2}{\partial k_1}(\mathbf{p}_{+,0}) \right] \sum_{\mathbf{x},\omega} \psi_{\mathbf{x},\omega}^+ \partial_{\omega} \psi_{\mathbf{x},\omega}^- \quad (15)$$

where  $\partial_{\omega}$  is the Fourier transform of  $ik_0 - \omega k_1$ . Again by (13), the prefactors in (14) and (15) are real. Instead, there are no local, relevant terms: the only possible one, a quadratic terms without derivatives, i.e. a *mass term*, cannot be generated by conservation of the total momentum. These facts imply that (9) (with parameter  $\lambda$ ) equals the massless Thirring model with parameter  $\lambda_T$ , up to terms which are irrelevant and thus cannot break down (4) (although they do determine the functional relationship between  $\lambda$  and  $\lambda_T$ ).

## V. CONCLUSION

We have showed that the IFP provides a detailed theoretical account of the numerical findings of [11] -plus some new predictions- in the case of weak interaction. The method should also work, with possibly different outcomes, for triangular and hexagonal lattices; and for two or more interacting copies of dimer models. Besides, the IFP should be applicable to the Six-Vertex model, which is equivalent to dimers on a square lattice with a staggered interaction. Including the results on Ashkin-Teller, Eight-Vertex and XYZ quantum chain, [23], the IFP seems quite an effective way for dealing with two dimensional, lattice, critical models with central charge  $c = 1$ . It might be possible that the IFP be applicable to  $c < 1$  models (for example through ideas in [15]).

## VI. ADDED NOTE: STRONG INTERACTION

In the opposite case of *strong* dimer interaction, i.e.  $|\lambda| \gg 1$ , the numerical findings of [11] indicate the existence of four different “columnar” phases. A theoretical justification of this case is also possible; however, as opposed to the weak interaction case, the outcome crucially depends on the choice of the interaction  $v(d, d')$ : for definiteness, we only discuss here the choice in [11], which assign an energy  $-\lambda < 0$  per each plaquette with one of the two dimer arrangements in Fig. 2.

Decorate a dimer configuration with wiggly lines as indicated in the same figure. Declare a plaquette to be “good” (or “ground state plaquette”) if it contains 2 wiggly lines; otherwise, declare the plaquette to be “bad” (or “excitation plaquettes”). Note that: a) dimer configurations on nearest neighbor good plaquettes must correspond to the same columnar ground state; b) the probability of a bad plaquettes occurrence is damped by a factor  $e^{-\frac{\lambda}{4}}$ , or smaller, per plaquette. Therefore, for  $\lambda$  positive and large, one can apply the standard Peierls’

argument to prove the existence of four different phases (see [33, 34] or the review [35]).

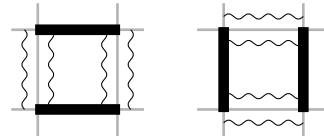


FIG. 2: For each plaquette containing two facing dimers draw four wiggly lines. An equivalent, but completely local way of evaluating the total energy of a dimer configuration is by assigning an energy of  $-\frac{\lambda}{4}$  per wiggly line.

## VII. ACKNOWLEDGMENTS

The basic ideas for dealing with the case of weak interaction and the case of strong interaction arose in several discussions with T. Spencer and M. Biskup, respectively.

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